JOURNAL OF APPROXIMATION THEORY 83, 220–237 (1995)

Reconstruction of Functions on the Basis of Sequences of Linear Functionals

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Communicated by T. J. Rivlin

Received December 2, 1992; accepted in revised form May 27, 1994

The question about a reconstruction of functions from a certain class is studied. The reconstruction is realised on the basis of sequences of linear functionals $l_n(f)_{n=1}^{\infty}$ of the form $l_n(f)=\sum_{k=0}^{m(n)} a_{nk} f(x_{nk})$. An explicit expression of the reconstructed function is given.

1. INTRODUCTION

Every function continuous on $[a, b]$ is uniquely determined by its values at a sequence of points $\{x_i\}_{i=0}^{\infty}$ which are dense in $[a, b]$. This is a trivial example showing that there is a sequence of linear functionals, namely $l_n(f)=f(x_n)$, $n=0, 1, \ldots$, which presents complete information about $f \in C[a, b]$. In particular, $l_n(f)=0$ for each n implies $f \equiv 0$.

Clearly the sequence of divided differences $f[x_0, x_1, \ldots, x_n]$, $n=0, 1, \ldots$ has the same property since the conditions

$$f[x_0]=f[x_0, x_1]=\cdots=f[x_0, x_1, \ldots, x_n]=0$$

are equivalent to $f(x_0)=f(x_1)=\cdots=f(x_n)=0$.

Now consider the next more general problem. Suppose that

$$\mathbf{X} := \{(x_{n0}, x_{n1}, \ldots, x_{nn}),\ n=0, 1, \ldots\}$$

is a given triangular matrix of points in $[a, b]$ such that $\max_k |x_{n,k+1}-x_{nk}| \to 0$ as $n \to \infty$. Suppose that $f[x_{n0}; x_{n1}, \ldots, x_{nn}]=0$ for each n and f from $C[a, b]$. Does this imply $f \equiv 0$? The question was studied by Eterman [3] in the case $\{x_{nk}\}_{k=0}^{n}$ are the extremal points of the Chebyshev polynomial $T_n(x)$ (i.e. $x_{nk}=\cos(k\pi/n)$). He proved that $f \equiv 0$, provided it has an absolutely convergent Fourier-Chebyshev series. The case $[a, b]=[0, 1]$, $x_{nk}=k/n$ is a well-known open problem in approximation theory.

* Supported by the Bulgarian Ministry of Education and Science under Grant MM-15.

0021-9045/95 $12.00

Note that the condition $f[x_{n0}, x_{n1}, ..., x_{nn}] = 0$ is equivalent to $e_n(f) = 0$, where $e_n(f)$ is the error of the best uniform approximation of f on the discrete set $\{x_{n0}, x_{n1}, ..., x_{nn}\}$ by polynomials from π_{n-1} (π_m denotes the set of all algebraic polynomials of degree m). Thus, this observation gives another interesting interpretation of our problem.

The divided difference $f[x_{n0}, x_{n1}, ..., x_{nn}]$ is a linear combination of the function values $f(x_{n0}), f(x_{n1}), ..., f(x_{nn})$. We study here sequences of linear functionals of the form

$$l_n(f) = \sum_{k=0}^{m(n)} a_{nk} f(x_{nk}), \qquad n = 1, 2, ...$$

and show some new examples of $\{a_{nk}\}$, $\{x_{nk}\}$ which have the property that $l_n(f) = 0$ implies $f \equiv 0$. Moreover, we give an explicit expression of f on the basis of the information $\{l_n(f)\}_{n=1}^{\infty}$.

2. Construction of the Functionals

Let us denote, as usual, by $T_n(x)$ and $U_n(x)$ the Chebyshev polynomials of the first and second kind, respectively.

Recall also their generating functions

$$T(x, t) = \sum_{n=0}^{\infty} T_n(x)\, t^n = \frac{1 - tx}{1 - 2tx + t^2}, \qquad x \in [-1, 1], \qquad |t| < 1;$$

$$U(x, t) = \sum_{n=0}^{\infty} U_n(x)\, t^n = \frac{1}{1 - 2tx + t^2}, \qquad x \in [-1, 1], \qquad |t| < 1.$$

With every function $f \in \mathbf{C}[-1, 1]$ we associate it expansions

$$\sum_{n=0}^{\infty} A_n(f)\, T_n(x), \tag{1}$$

where

$$A_0 = A_0(f) = \frac{1}{\pi} \int_{-1}^{1} (1 - x^2)^{-1/2} f(x)\, dx,$$

$$A_n = A_n(f) = \frac{2}{\pi} \int_{-1}^{1} (1 - x^2)^{-1/2} f(x)\, T_n(x)\, dx, \qquad n = 1, 2, ...$$

and

$$\sum_{n=0}^{\infty} B_n(f)\, U_n(x), \tag{2}$$

where

$$B_n = B_n(f) = \frac{2}{\pi} \int_{-1}^{1} (1-x^2)^{1/2} f(x)\, U_n(x)\, dx, \qquad n = 0, 1, \dots$$

Introduce the classes:

$$\mathbf{AT} := \left\{ f\colon \sum_{n=0}^{\infty} A_n T_n(x) \text{ is absolutely convergent} \right\},$$

$$\mathbf{AU} := \left\{ f\colon \sum_{n=0}^{\infty} B_n U_n(x) \text{ is absolutely convergent} \right\},$$

$$\mathbf{AT}_\varepsilon := \{ f\colon A_n = O(n^{-1-\varepsilon}) \},$$

$$\mathbf{AU}_\varepsilon := \{ f\colon B_n = O(n^{-1-\varepsilon}) \}.$$

We study here the linear functionals

$$E_n(f) = \frac{1}{n} \sum_{m=0}^{n}{}^{*} (-1)^m f\left(\cos \frac{m\pi}{n}\right), \qquad n = 1, 2, \dots \tag{3}$$

$$I_n(f) = \frac{1}{n} \sum_{m=0}^{n-1}{}^{*} (-1)^m \sin \frac{(2m+1)\pi}{2n} f\left(\cos \frac{(2m+1)\pi}{2n}\right), \qquad n = 1, 2, \dots \tag{4}$$

$$L_n(f) = \frac{2}{n} \sum_{m=0}^{[n/2]}{}^{*} f\left(\cos \frac{2m\pi}{n}\right), \qquad n = 1, 2, \dots \tag{5}$$

$$M_n(f) = \frac{2}{n} \sum_{m=0}^{[(n-1)/2]}{}^{*} f\left(\cos \frac{(2m+1)\pi}{n}\right), \qquad n = 1, 2, \dots \tag{6}$$

$$P_{s,n}(f) = \frac{1}{sn} \sum_{m=0}^{[sn/2]}{}^{*} a_m f\left(\cos \frac{2m\pi}{sn}\right), \tag{7}$$

where $s \geqslant 2$ is a positive integer,

$$a_m = \begin{cases} 2(s-1), & \text{if } s \mid m \\ -2, & \text{if } s \nmid m \end{cases}, \qquad n = 1, 2, \dots$$

$$R_{s,n}(f) = \frac{4}{sn} \sum_{m=0}^{[sn/2]}{}^{*} \cos \frac{2m\pi}{s} f\left(\cos \frac{2m\pi}{sn}\right), \tag{8}$$

where $s \geqslant 2$ is a positive integer, $n = 1, 2, \dots$

The asterisk on the summation sign means that the terms with $f(1)$ or $f(-1)$ are to be halved. We shall also use the notation $\sum^{S}$, which means that the summation index skips all multiples of s.

Theorem 1. *If $f \in \mathbf{AT}$, $g \in \mathbf{AU}$, then*

$$E_n(f) = \sum_{m=0}^{\infty} A_{(2m+1)n}, \tag{9}$$

$$I_n(g) = \sum_{m=0}^{\infty} (-1)^m B_{(2m+1)n-1}, \tag{10}$$

$$L_n(f) = \sum_{m=0}^{\infty} A_{mn}, \tag{11}$$

$$M_n(f) = \sum_{m=0}^{\infty} (-1)^m A_{mn}, \tag{12}$$

$$P_{s,n}(f) = \sum_{m=1}^{\infty}{}^{s} A_{mn}, \tag{13}$$

$$R_{s,n}(f) = A_n + \sum_{m=1}^{\infty} A_{(ms \pm 1)n}. \tag{14}$$

Proof. Some of these relations are known. See, for example, a simple proof of (9) in [6, p. 93 and p. 174]. We use here another new approach which makes it possible to establish relations of this kind. We give a detailed proof of (9) only.

Denote by u and v the zeros of the polynomial $p(t) = t^2 - 2tx + 1$, where x is a parameter from $[-1, 1]$. Then $T(x, t)$ may be written in the form

$$T(x, t) = -\frac{1}{2}\left(\frac{u}{t-u} + \frac{v}{t-v}\right).$$

After the transformation $x = \cos\theta$, $\theta \in [0, \pi]$ the zeros become $u = \cos\theta + i\sin\theta$ and $v = \cos\theta - i\sin\theta$. Denote

$$x_m := \cos\frac{m\pi}{n}, \qquad u_m := \cos\frac{m\pi}{n} + i\sin\frac{m\pi}{n}, \qquad v_m := \cos\frac{m\pi}{n} - i\sin\frac{m\pi}{n}.$$

Apply first the functional E_n to the function $T(\cdot, t)$.

$$\begin{aligned}
E_n(T(\cdot, t)) &= \frac{1}{n}\sum_{m=0}^{n}{}^{*} (-1)^m T(x_m, t) \\
&= \frac{1}{n}\sum_{m=1}^{n-1} (-1)^m T(x_m, t) + \frac{1}{2n}(T(x_0, t) + (-1)^n T(x_n, t)) \\
&= -\frac{1}{2n}\sum_{m=1}^{n-1} (-1)^m \left(\frac{u_m}{t-u_m} + \frac{v_m}{t-v_m}\right) + \frac{1}{2n}\left(\frac{1}{1-t} + \frac{(-1)^n}{1+t}\right) \\
&= -\frac{1}{2n}\sum_{m=0}^{2n-1} (-1)^m \frac{u_m}{t-u_m},
\end{aligned}$$

because $v_m = u_{2n-m}$. The last sum is a rational function with a denominator $t^{2n}-1$, because $\{u_m\}_{m=0}^{2n-1}$ are the $2n$th roots of the unity. Denote by $\varphi(t)$ the numerator of this rational function. Clearly $\varphi \in \pi_{2n-1}$. We have

$$\frac{\varphi(t)}{t^{2n}-1} = -\frac{1}{2n}\sum_{m=0}^{2n-1}(-1)^m \frac{u_m}{t-u_m}$$

and therefore

$$\varphi(t) = -\frac{1}{2n}\sum_{m=0}^{2n-1}(-1)^m u_m \frac{t^{2n}-1}{t-u_m}.$$

In particular, for $t = u_l$

$$\begin{aligned}\varphi(u_l) &= -\frac{1}{2n}(-1)^l u_l \left.\frac{t^{2n}-1}{t-u_l}\right|_{t=u_l} \\ &= -\frac{1}{2n}(-1)^l u_l(2n)\, u_l^{2n-1} = (-1)^{l+1} = -u_l^n,\end{aligned}$$

$$l = 0, 1, ..., 2m-1.$$

Since $\varphi(t)$ is determined uniquely by its values at $\{u_l\}_{l=0}^{2n-1}$, we obtain $\varphi(t) = -t^n$. Hence

$$E_n(T(\cdot, t)) = \frac{t^n}{1-t^{2n}} = \sum_{m=0}^{\infty} t^{(2m+1)n}.$$

On the other hand, since E_n is bounded and therefore continuous functional, it follows from the expression of the generating function of $T_n(x)$ that

$$E_n(T(\cdot, t)) = \sum_{j=0}^{\infty} t^j E_n(T_j(x)).$$

Therefore

$$E_n(T_j(x)) = \begin{cases} 1, & \text{if } j = (2m+1)\,n; \\ 0, & \text{otherwise.}\end{cases}$$

Finally, applying E_n to (1) we get

$$E_n(f) = E_n\left(\sum_{k=0}^{\infty} A_k T_k(x)\right) = \sum_{k=0}^{\infty} A_k E_n(T_k(x)) = \sum_{m=0}^{\infty} A_{(2m+1)n}.$$

3. Main Results

We solve here the reconstruction problem on the basis of the functionals (3)–(7). Our method uses the properties of the Möbius function $\mu(n)$. Let us recall that $\mu(n)$ is defined in the following way:

$$\mu(1)=1;$$
$$\mu(n)=0, \quad \text{if} \quad p^2 \mid n, \quad p \text{ is prime};$$
$$\mu(n)=(-1)^m, \quad \text{if} \quad n=p_1 p_2 \cdots p_m,$$
$$p_i \text{ are distinct primes}, \quad i=1, 2, ..., m.$$

It is known that

$$\sum_{d \mid n} \mu(d)=\begin{cases}1, & \text{if } n=1; \\ 0, & \text{otherwise.}\end{cases}$$

The proof may be found in [7, p. 27].

Introduce the arithmetic function ν by the equalities

$$\nu(1)=-1;$$
$$\sum_{d \mid n} \nu(d)(-1)^{n/d}=0, \quad n=2, 3, ...$$

For each positive integer n denote

$$a(n) := \text{the highest power of 2, which divides } n,$$
$$r(n) := \frac{n}{2^{a(n)}}.$$

A simple consequence from [1, Lemma 6] is that

$$\nu(n)=-2^{a(n)-1}\mu(r(n)).$$

(under the convention that $2^{-1}=1$).

THEOREM 2. *Let $f \in \mathbf{AT}_\varepsilon$. The functionals $\{L_n(f)\}_{n=1}^{\infty}$ and A_0 determine f uniquely. Moreover*

$$f(x)=A_0+\sum_{j=1}^{\infty}(L_j(f)-A_0)\sum_{k \mid j}\mu(k)\, T_{j/k}(x). \tag{15}$$

Proof. Putting $L_n := L_n(f)$ and $L_{n0} := L_n - A_0$, for convenience, one may write (11) as

$$L_{n0} = \sum_{m=1}^{\infty} A_{mn}, \qquad n = 1, 2, ..., \tag{16}$$

which is an infinite linear system with respect to $\{A_n\}_{n=1}^{\infty}$. We shall show that the homogeneous system $\sum_{m=1}^{\infty} A_{mn} = 0$ admits only the trivial solution.

Let $M = p_1 p_2 \cdots p_l$ be the product of the first l primes. Then,

$$0 = \sum_{d \mid M} \mu(d) \sum_{m=1}^{\infty} A_{m\,dn} = \sum_{j=1}^{\infty} A_{jn} \sum_{\substack{d \mid M \\ d \mid j}} \mu(d)$$

$$= \sum_{j=1}^{\infty} A_{jn} \sum_{\substack{d \mid \delta \\ \delta = (j, M)}} \mu(d) = \sum_{\substack{j=1 \\ (j, M) = 1}}^{\infty} A_{jn}$$

(we used the change $md = j$ in the second equality).
Hence

$$0 = A_n + \sum_{\substack{j = p_{l+1} \\ (j, M) = 1}}^{\infty} A_{jn},$$

and consequently

$$|A_n| \leqslant \sum_{\substack{j = p_{l+1} \\ (j, M) = 1}}^{\infty} |A_{jn}| \leqslant \sum_{j = p_{l+1}}^{\infty} |A_j|.$$

Letting $l \to \infty$ we conclude that $A_n = 0$.

So if the system (16) has a solution it must be unique. We prove next that this solution is given by the expression

$$A_n = \sum_{k=1}^{\infty} \mu(k)\, L_{kn0}.$$

Note first that the series is absolutely convergent. Indeed,

$$\sum_{k=1}^{\infty} |\mu(k)|\, |L_{kn0}| \leqslant \sum_{k=1}^{\infty} |L_{kn0}|$$

and since

$$|L_{kn0}| \leqslant \sum_{m=1}^{\infty} |A_{mkn}| \leqslant C_1 \sum_{m=1}^{\infty} \frac{1}{(mkn)^{1+\varepsilon}} = C_2 \frac{1}{(kn)^{1+\varepsilon}},$$

with some constants C_1, C_2, the series is evidently convergent. Now inserting the expression of A_n in (16) we get

$$\sum_{m=1}^{\infty} \sum_{k=1}^{\infty} \mu(k) L_{knm0} = \sum_{j=1}^{\infty} L_{jn0} \sum_{k \mid j} \mu(k) = L_{n0},$$

which shows that it is a solution. This proof will be complete if we show that the change of the order of the summation is correct. But this is a consequence of the fact that any of the series in the last equality is absolutely convergent. Let us prove, for example, that $\sum_{j=1}^{\infty} L_{jn0} \sum_{k \mid j} \mu(k)$ is absolutely convergent.

$$\begin{aligned}\sum_{j=1}^{\infty} |L_{jn0}| \left|\sum_{k \mid j} \mu(k)\right| &\leqslant \sum_{j=1}^{\infty} |L_{jn0}| \sum_{k \mid j} |\mu(k)| \leqslant \sum_{j=1}^{\infty} |L_{jn0}|\, \tau(j) \\ &\leqslant \sum_{j=1}^{\infty} |L_{jn0}|\, \tau(nj) \leqslant \sum_{j=1}^{\infty} |L_{j0}|\, \tau(j) \\ &\leqslant C_1 \sum_{j=1}^{\infty} \frac{\tau(j)}{j^{1+\varepsilon}} = C_1 \sum_{j=1}^{\infty} \frac{1}{j^{1+\varepsilon/2}} \cdot \frac{\tau(j)}{j^{\varepsilon/2}} \\ &\leqslant C_2 \sum_{j=1}^{\infty} \frac{1}{j^{1+\varepsilon/2}}.\end{aligned}$$

Here we denote by $\tau(n)$ the number of the divisors of n and we also use the asymptotic equality $\tau(j) = O(j^{\delta})$, which holds for any positive integer j and any $\delta > 0$ (see [7, p. 34]). C_1 and C_2 are constants.

Using the presentation (1) we find

$$\begin{aligned}f(x) &= \sum_{n=0}^{\infty} A_n T_n(x) = A_0 + \sum_{n=1}^{\infty} \left(\sum_{k=1}^{\infty} \mu(k) L_{kn0} \right) T_n(x) \\ &= A_0 + \sum_{j=1}^{\infty} L_{j0} \sum_{k \mid j} \mu(k)\, T_{j/k}(x).\end{aligned}$$

The proof is complete.

THEOREM 3. *Let $f \in \mathbf{AT}_{\varepsilon}$. The functionals $\{M_n(f)\}_{n=1}^{\infty}$ and A_0 determine f uniquely. Moreover*

$$f(x) = A_0 + \sum_{j=1}^{\infty} (M_j(f) - A_0) \sum_{k \mid j} \nu(k)\, T_{j/k}(x). \tag{17}$$

Proof. Putting $M_n := M_n(f)$ and $M_{n0} := M_n - A_0$, for convenience, one may write (12) as

$$M_{n0} = \sum_{m=1}^{\infty} (-1)^m A_{mn}, \qquad n = 1, 2, \ldots \tag{18}$$

We shall show that the homogeneous system $\sum_{m=1}^{\infty}(-1)^m A_{mn}=0$ admits only the trivial solution.

Let M be the least common multiple of the first l positive integers. Then

$$0=\sum_{d\mid M}\nu(d)\sum_{m=1}^{\infty}(-1)^m A_{m\,dn}=\sum_{j=1}^{\infty}A_{jn}\sum_{\substack{d\mid M\\ d\mid j}}\nu(d)(-1)^{j/d}$$

$$=\sum_{j=1}^{l}A_{jn}\sum_{\substack{d\mid M\\ d\mid j}}\nu(d)(-1)^{j/d}+\sum_{j=l+1}^{\infty}A_{jn}\sum_{\substack{d\mid M\\ d\mid j}}\nu(d)(-1)^{j/d}$$

$$=\sum_{j=1}^{l}A_{jn}\sum_{d\mid j}\nu(d)(-1)^{j/d}+\sum_{j=l+1}^{\infty}A_{jn}\sum_{\substack{d\mid M\\ d\mid j}}\nu(d)(-1)^{j/d}$$

$$=A_n+\sum_{j=l+1}^{\infty}A_{jn}\sum_{\substack{d\mid M\\ d\mid j}}\nu(d)(-1)^{j/d}.$$

Hence

$$|A_n|=\left|\sum_{j=l+1}^{\infty}A_{jn}\sum_{\substack{d\mid M\\ d\mid j}}\nu(d)(-1)^{j/d}\right|.$$

But

$$\left|\sum_{\substack{d\mid M\\ d\mid j}}\nu(d)(-1)^{j/d}\right|\leqslant\sum_{\substack{d\mid M\\ d\mid j}}|\nu(d)|\leqslant\sum_{d\mid j}|\nu(d)|=\sum_{\substack{d\mid j\\ j=2^{a(j)}r(j)}}|\nu(d)|$$

$$=\sum_{c=0}^{a(j)}\sum_{d\mid r(j)}|\nu(2^c d)|=\sum_{c=0}^{a(j)}2^{c-1}\sum_{d\mid r(j)}|\mu(d)|$$

$$\leqslant 2^{a(j)}\tau(r(j))$$

and therefore

$$|A_n|\leqslant\sum_{j=l+1}^{\infty}2^{a(j)}\tau(r(j))\,|A_{jn}|$$

$$\leqslant C_1\sum_{j=l+1}^{\infty}2^{a(j)}\tau(r(j))\frac{1}{(2^{a(j)}r(j)\,n)^{1+\varepsilon}}$$

$$=\frac{C_1}{n^{1+\varepsilon}}\sum_{j=l+1}^{\infty}2^{-\varepsilon a(j)}\frac{\tau(r(j))}{r(j)^{1+\varepsilon}}.$$

Because of the inequality

$$\sum_{j=1}^{\infty} 2^{-\varepsilon a(j)} \frac{\tau(r(j))}{r(j)^{1+\varepsilon}} \leqslant \sum_{a=0}^{\infty} 2^{-\varepsilon a} \sum_{r=1}^{\infty} \frac{\tau(r)}{r^{1+\varepsilon}}$$

the series $\sum_{j=1}^{\infty} 2^{-\varepsilon a(j)}\tau(r(j))/r(j)^{1+\varepsilon}$ is convergent.

Letting $l \to \infty$ we obtain that $A_n = 0$.

So, if the system (18) has a solution it must be unique. We prove next that this solution is given by the expression $A_n = \sum_{k=1}^{\infty} \nu(k)\, M_{kn0}$. As in the previous case we show that this series is absolutely convergent. Indeed we have

$$|M_{kn0}| \leqslant \sum_{m=1}^{\infty} |(-1)^m A_{mkn}| = \sum_{m=1}^{\infty} |A_{mkn}| = \frac{C_1}{(kn)^{1+\varepsilon}}.$$

Thus

$$\begin{aligned}
\sum_{k=1}^{\infty} |\nu(k)\, M_{kn0}| &= \sum_{k=1}^{\infty} 2^{a(k)-1}\, |\mu(r(k))|\, |M_{kn0}| \\
&\leqslant \frac{C_1}{n^{1+\varepsilon}} \sum_{k=1}^{\infty} 2^{a(k)-1} \frac{1}{(2^{a(k)} r(k))^{1+\varepsilon}} \\
&= \frac{C_1}{n^{1+\varepsilon}} \sum_{k=1}^{\infty} 2^{-\varepsilon a(k)-1} \frac{1}{(r(k))^{1+\varepsilon}} \\
&\leqslant \frac{C_2}{n^{1+\varepsilon}} \sum_{a=0}^{\infty} 2^{-\varepsilon a} \sum_{r=1}^{\infty} \frac{1}{r^{1+\varepsilon}}
\end{aligned}$$

which yields that $\sum_{k=1}^{\infty} \nu(k)\, M_{kn0}$ is absolutely convergent. Now inserting $\sum_{k=1}^{\infty} \nu(k)\, M_{kn0}$ in (18) we get

$$\sum_{m=1}^{\infty} (-1)^m \sum_{k=1}^{\infty} \nu(k)\, M_{kmn0} = \sum_{j=1}^{\infty} M_{jn0} \sum_{k \mid j} \nu(k)(-1)^{j/k} = M_{n0}.$$

In order to complete the proof we should show that the change of the order of the summation is correct. Let us establish the absolute convergence of the series $\sum_{j=0}^{\infty} M_{jn0} \sum_{k \mid j} \nu(k)(-1)^{j/k}$, for example.

$$\begin{aligned}
\sum_{j=1}^{\infty} |M_{jn0}| \left|\sum_{k \mid j} \nu(k)(-1)^{j/k}\right| &\leqslant \sum_{j=1}^{\infty} |M_{jn0}| \sum_{k \mid j} |\nu(k)| \\
&\leqslant \sum_{j=1}^{\infty} |M_{jn0}|\, 2^{a(j)} \tau(r(j)) \\
&\leqslant \frac{C_3}{n^{1+\varepsilon}} \sum_{j=1}^{\infty} 2^{-\varepsilon a(j)} \frac{\tau(r(j))}{r(j)^{1+\varepsilon}}.
\end{aligned}$$

The convergence of the last series was already proved.

Using the presentation (1) we find

$$f(x)=\sum_{n=0}^{\infty} A_n T_n(x)=A_0+\sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty} \nu(k) M_{kn0}\right) T_n(x)$$

$$=A_0+\sum_{j=1}^{\infty} M_{j0} \sum_{k \mid j} \nu(k) T_{j/k}(x).$$

The proof is complete.

Note, also, that considering separately the cases of even and odd n, one can obtain other functionals, as it is done in [1], [3], [5]. For example, the functional $M_{2n}(f)$ is used for the reconstruction of an even function (see [1, Theorem 4]). Following the proof of Theorem 1, one may get the next known results.

THEOREM A. *Let $f \in \mathbf{AT}_\varepsilon$. The functionals $\{E_n(f)\}_{n=1}^{\infty}$ and $f(1)$ determine f uniquely.*

THEOREM B. *Let $f \in \mathbf{AU}_\varepsilon$. The functionals $\{I_n(f)\}_{n=1}^{\infty}$ and $f(1)$ determine f uniquely.*

We mention these theorems only for completeness. Their proofs may be found in [2], [3], [4].

Next we use a family of functionals, namely $\{P_{s,n}(f)\}_{n=1}^{\infty}$ for the reconstruction of the function.

THEOREM 4. *Let $f \in \mathbf{AT}_\varepsilon$ and $s \geqslant 2$ be a fixed prime number. The functionals $\{P_{s,n}(f)\}_{n=1}^{\infty}$ and $f(1)$ determine f uniquely. Moreover*

$$f(x)=f(1)+\sum_{j=1}^{\infty} P_{s,j}(f) \sum_{\substack{k \mid j \\ s \nmid k}} \mu(k)(T_{j/k}(x)-1). \tag{19}$$

Proof. Consider the system of equations

$$P_{s,n}=\sum_{m=1}^{\infty}{}^{s} A_{nm}, \qquad (P_{s,n}=P_{s,n}(f)).$$

We shall show that the corresponding homogeneous system has only the trivial solution. In order to do this, consider the infinite sequence $\mathbf{P}$ of all prime numbers, from which s is removed.

Let $M = p_1 p_2 \cdots p_l$ be the product of the first l primes in $\mathbf{P}$. We have

$$0 = \sum_{d \mid M} \mu(d) \sum_{m=1}^{\infty}{}^{S} A_{m\,dn} = \sum_{j=1}^{\infty}{}^{S} A_{jn} \sum_{\substack{d \mid M \\ d \mid j}} \mu(d)$$

$$= \sum_{j=1}^{\infty}{}^{S} A_{jn} \sum_{\substack{d \mid \delta \\ \delta = (j, M)}} \mu(d) = \sum_{\substack{j=1 \\ (j, M) = 1}}^{\infty}{}^{S} A_{jn}.$$

Further, the proof follows that one of Theorem 2. Here the solution of the system is obtained in the form

$$A_n = \sum_{k=1}^{\infty}{}^{S} \mu(k)\, P_{s,\,kn}.$$

Now, using (1) we get

$$f(x) = \sum_{n=0}^{\infty} A_n T_n(x) = A_0 + \sum_{n=1}^{\infty} \left(\sum_{k=1}^{\infty}{}^{S} \mu(k)\, P_{s,\,kn} \right) T_n(x)$$

$$= A_0 + \sum_{j=1}^{\infty} P_{s,\,j} \sum_{\substack{k \mid j \\ s \nmid k}} \mu(k)\, T_{j/k}(x).$$

On the other hand

$$f(1) = A_0 + \sum_{n=1}^{\infty} A_n = A_0 + \sum_{j=1}^{\infty} P_{s,\,j} \sum_{\substack{k \mid j \\ s \nmid k}} \mu(k)$$

and hence

$$f(x) = f(1) + \sum_{j=1}^{\infty} P_{s,\,j}(f) \sum_{\substack{k \mid j \\ s \nmid k}} \mu(k)\, (T_{j/k}(x) - 1).$$

The proof is complete.

Next we need some new notations and auxiliary propositions.

Let $s \geq 2$ be a fixed positive integer. Introduce the following set of positive integers

$$\mathbf{K}_s := \left\{ \{ks \pm 1\}_{k=1}^{\infty} \cup \{1\} \right\}.$$

In other words $\mathbf{K}_s$ is a union of two arithmetic progressions with difference s. It is not difficult to see that $\mathbf{K}_s$ is closed with respect to multiplication, i.e. the product of two numbers from $\mathbf{K}_s$ is also its element. For convenience we shall mark m by $\tilde{}$ (i.e. we shall write $\tilde{m}$) to denote that the positive integer m (which may be presented as $m = ks \pm 1$ for some positive integer k) is considered as an element of $\mathbf{K}_s$.

We say also that $\tilde{b}$ divides $\tilde{a}$ (and denote this by $\tilde{b} \mid \tilde{a}$) if there exists $\tilde{c}$, such that $\tilde{a} = \tilde{b}\tilde{c}$ (Further, $\tilde{c}$ would be denoted by $\tilde{a}/\tilde{b}$).

It is easy to see that if $\tilde{a} \in \mathbf{K}_s$, $\tilde{b} \in \mathbf{K}_s$ and $a/b \in \mathbf{Z}$, then $\tilde{b} \mid \tilde{a}$, i.e. $\tilde{a}/\tilde{b} \in \mathbf{K}_s$ and $\tilde{a}/\tilde{b} = a/b$. Further, for the elements of $\mathbf{K}_s$, we shall consider operation "division" only in $\mathbf{K}_s$.

We call "prime" in $\mathbf{K}_s$ any element $\tilde{m}$ which has only two divisors from $\mathbf{K}_s$ (namely $\tilde{1}$ and $\tilde{m}$). The rest elements (except $\tilde{1}$) are said to be "composite numbers".

For example:

$$\mathbf{K}_5 = \{\tilde{1}, \tilde{4}, \tilde{6}, \tilde{9}, \widetilde{11}, \widetilde{14}, \widetilde{16}, \widetilde{19}, \widetilde{21}, \widetilde{24}, \widetilde{26}, \ldots\}.$$

The elements $\tilde{4}, \tilde{6}, \tilde{9}, \widetilde{11}, \widetilde{14}, \widetilde{19}, \widetilde{21}, \widetilde{26}$ have only two divisors. The first composite number is $\widetilde{16} = \tilde{4}^2$, and the first composite with different prime divisors is $\widetilde{24} = \tilde{4} \times \tilde{6}$. Introduce the arithmetic function $\tilde{\mu}$, defined in $\mathbf{K}_s$ by he equalities

$$\tilde{\mu}(\tilde{1}) = 11;$$

$$\sum_{\tilde{d} \mid \tilde{n}} \tilde{\mu}(\tilde{d}) = 0, \qquad \tilde{n} > \tilde{1}.$$

LEMMA 1. *Let a be a positive integer and $a \equiv \pm 1 \pmod 6$. If $a = bc$, then $b \equiv \pm 1 \pmod 6$ and $c \equiv \pm 1 \pmod 6$.*

Proof. Let $b = 6b_1 + r_1$ and $c = 6c_1 + r_2$, $0 \leqslant r_1 \leqslant 5$, $0 \leqslant r_2 \leqslant 5$. It is seen from the table that from all combinations for the product $r_1 r_2$, the above-mentioned congruence is true only when $r_1 \equiv \pm 1 \pmod 6$ and $r_2 \equiv \pm 1 \pmod 6$.

r_2 \ r_1:	1	2	3	4	5
1	1	2	3	4	5
2	2	4	0	2	4
3	3	0	3	0	3
4	4	2	0	4	2
5	5	4	3	2	1

The proof is complete.

Next we give a lemma which could be recognized as the Fundamental theorem of arithmetics in $\mathbf{K}_6$.

LEMMA 2. *Each $\tilde{a} \in \mathbf{K}_6$ has a unique representation as a product of primes up to the order of the factors*

$$\tilde{a} = \tilde{p}_1 \tilde{p}_2 \cdots \tilde{p}_m$$

(*some of these primes may be equal*).

Proof. Prove first the existence. Obviously the existence is clear for all primes of $\mathbf{K}_6$ (i.e. $\tilde{5}, \tilde{7}, \ldots$). Next we proceed by induction. Assume also that the existence holds for all elements of $\mathbf{K}_6$, which do not exceed $\tilde{u}$. Let $\tilde{v}$ be the next element of $\mathbf{K}_6$ (we assume that all elements of the set are ordered by size). We shall prove the existence of primes, whose product is $\tilde{v}$. Choose the least divisor of $\tilde{v}$ from the sequence $\tilde{5}, \tilde{7}, \ldots, \tilde{u}, \tilde{v}$. If it is $\tilde{v}$ then $\tilde{v}$ is prime. Otherwise we have $\tilde{v} = \tilde{p}\tilde{q}$, where $\tilde{p}$ is prime and obviously $\tilde{q} \leqslant \tilde{u}$. Then according to the induction hypothesis $\tilde{q}$ and consequently $\tilde{v}$ may be presented as a product of primes.

Now we shall prove the uniqueness of the representation (up to the order of the factors). Suppose that for some $\tilde{a} \in \mathbf{K}_6$

$$\tilde{a} = \tilde{p}_1 \tilde{p}_2 \cdots \tilde{p}_m = \tilde{q}_1 \tilde{q}_2 \cdots \tilde{q}_n,$$

where $\tilde{p}_i$, $\tilde{q}_j$-are primes in $\mathbf{K}_6$. Consider p_1 and suppose that it is composite in N. Then $p_1 = p'_1 p''_1$ and we derive from Lemma 1 that $\tilde{p}'_1 \in \mathbf{K}_6$ and $\tilde{p}''_1 \in \mathbf{K}_6$. This contradicts our assumption that $\tilde{p}_1$ is prime in $\mathbf{K}_6$. Hence p_1 is prime in N. Using similar arguments we may establish that each of the numbers p_i, q_j is prime in N. Then all p_i and q_j in the equality

$$p_1 p_2 \cdots p_m = q_1 q_2 \cdots q_n$$

are primes and from the Fundamental theorem of arithmetics we derive the uniqueness (up to the order of the factors).

LEMMA 3. *For the function $\tilde{\mu}(\tilde{n})$, defined in $\mathbf{K}_6$ the following equalities are true*:

$\tilde{\mu}(\tilde{1}) = 1;$

$\tilde{\mu}(\tilde{n}) = 0, \quad$ *if* $\tilde{p}^2 \mid \tilde{n}, \quad \tilde{p}$ *is prime*;

$\tilde{\mu}(\tilde{n}) = (-1)^m, \quad$ *if* $\tilde{n} = \tilde{p}_1 \tilde{p}_2 \cdots \tilde{p}_m, \quad \tilde{p}_i$ *are distinct primes.*

Proof. We may conclude from Lemma 2 that for each $\tilde{n} \in \mathbf{K}_6$ there exists a unique representation of the form

$$\tilde{n} = \tilde{p}_1^{\alpha_1} \tilde{p}_2^{\alpha_2} \cdots \tilde{p}_m^{\alpha_m},$$

where $\tilde{p}_i$, $i=1, 2, \ldots, m$ are primes, $1 \leqslant a_i$. Here a_i, $i=1, 2, \ldots, m$, are the multiplicity of the corresponding prime divisors. The proof goes by induction on $\tilde{n}$. Obviously, the statement is true for $\tilde{1}$ and all primes. Suppose that it is true also for all numbers of $\mathbf{K}_6$, less than $\tilde{n}$. It is clear that any divisor $\tilde{d}$ of $\tilde{n}$ has the form

$$\tilde{d} = \tilde{p}_1^{b_1} \cdots \tilde{p}_m^{b_m}, \qquad 0 \leqslant b_i \leqslant a_i.$$

Note, also, that $\tilde{\mu}(\tilde{d}) \neq 0$ only when $b_i = 1$ for $i=1, 2, \ldots, m$. Then

$$\tilde{\mu}(\tilde{n}) = -\sum_{\substack{\tilde{d} \mid \tilde{n} \\ \tilde{d} < \tilde{n}}} \tilde{\mu}(\tilde{d}) = -\left(\binom{m}{0} - \binom{m}{1} + \binom{m}{2} - \cdots \right.$$
$$\left. + (-1)^{m-1}\binom{m}{m-1} + \varepsilon_m\right),$$

where

$$\varepsilon_m = \begin{cases} (-1)^m, & \text{if at least one } a_i \geqslant 2; \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\tilde{\mu}(\tilde{n}) = \begin{cases} 0, & \text{if at least one } a_i \geqslant 2; \\ (-1)^m, & \text{otherwise.} \end{cases}$$

Denote by $\tilde{\tau}(\tilde{n})$ the number of divisors (in $\mathbf{K}_6$) of $\tilde{n}$. It is easy to see that $\tilde{\tau}(\tilde{n}) \leqslant \tau(n)$.

Further we shall consider only the case $s=6$, i.e. the set $\mathbf{K}_6$.

THEOREM 5. *Let* $f \in \mathbf{AT}_\varepsilon$. *The functionals* $\{R_{6,n}(f)\}_{n=1}^{\infty}$ *and* $f(1)$ *determine* f *uniquely. Moreover*

$$f(x) = f(1) + \sum_{j=1}^{\infty} R_{6,j}(f) \sum_{\tilde{k} \mid j} \tilde{\mu}(\tilde{k})(T_{j/k}(x) - 1). \tag{20}$$

(*Here the second sum is taken over all divisors of* j *which are elements of* $\mathbf{K}_6$).

Proof. Follow the proof of Theorem 2. From (14) we have the equality

$$R_{6,n} = \sum_{\tilde{m}=\tilde{1}}^{\infty} A_{\tilde{m}n}, \qquad (R_{6,n} = R_{6,n}(f)).$$

We shall show that the homogeneous system $\sum_{\tilde{m}=\tilde{1}}^{\infty} A_{\tilde{m}n}=0$ admits only the trivial solution.

Let $\tilde{M}=\tilde{p}_1\tilde{p}_2\cdots\tilde{p}_l$ be the product of the first l prime numbers of $\mathbf{K}_6$. Then

$$0=\sum_{\tilde{d}\mid\tilde{M}}\tilde{\mu}(\tilde{d})\sum_{\tilde{m}=\tilde{1}}^{\infty}A_{\tilde{m}\tilde{d}n}=\sum_{\tilde{j}=\tilde{1}}^{\infty}A_{\tilde{j}n}\sum_{\substack{\tilde{d}\mid\tilde{M}\\ \tilde{d}\mid\tilde{j}}}\tilde{\mu}(\tilde{d})$$

$$=\sum_{\tilde{j}=\tilde{1}}^{\infty}A_{\tilde{j}n}\sum_{\substack{d\mid\tilde{\delta}\\ \tilde{\delta}=(\tilde{j},\tilde{M})}}\tilde{\mu}(\tilde{d})=\sum_{\substack{\tilde{j}=\tilde{1}\\ (\tilde{j},\tilde{M})=\tilde{1}}}^{\infty}A_{\tilde{j}n}.$$

Hence

$$|A_n|\leqslant\sum_{\tilde{j}=\tilde{p}_{l+1}}^{\infty}|A_{\tilde{j}n}|\leqslant\sum_{j=p_{l+1}}^{\infty}|A_j|.$$

Letting $l\to\infty$ we derive that $A_n=0$. We used here that there are infinitely many primes (in $\mathbf{K}_6$). This is a consequence of Dirichlet's theorem for the prime numbers in arithmetic progression and from the fact that each prime (in $\mathbf{N}$), belonging to $\mathbf{K}_6$ is prime, also, for $\mathbf{K}_6$. We shall show that $\sum_{\tilde{k}=\tilde{1}}^{\infty}\tilde{\mu}(\tilde{k})\,R_{6,\tilde{k}n}$ is a solution of the system.

Since

$$|R_{6,\tilde{k}n}|=\left|\sum_{\tilde{m}=\tilde{1}}^{\infty}A_{\tilde{k}\tilde{m}n}\right|\leqslant\sum_{\tilde{m}=\tilde{1}}^{\infty}|A_{\tilde{k}\tilde{m}n}|\leqslant\sum_{m=1}^{\infty}|A_{kmn}|=O(kn)^{-1-\varepsilon},$$

then $\sum_{\tilde{k}=\tilde{1}}^{\infty}\tilde{\mu}(\tilde{k})\,R_{6,\tilde{k}n}$ is absolutely convergent. Further,

$$\sum_{\tilde{m}=\tilde{1}}^{\infty}\sum_{\tilde{k}=\tilde{1}}^{\infty}\tilde{\mu}(\tilde{k})\,R_{6,\tilde{k}\tilde{m}n}=\sum_{\tilde{j}=\tilde{1}}^{\infty}R_{6,\tilde{j}n}\sum_{\tilde{k}\mid\tilde{j}}\tilde{\mu}(\tilde{k})=R_{6,n}.$$

We need only show the absolute convergence of $\sum_{\tilde{j}=\tilde{1}}^{\infty}R_{6,\tilde{j}n}\sum_{\tilde{k}\mid\tilde{j}}\tilde{\mu}(\tilde{k})$. We have

$$\sum_{\tilde{j}=\tilde{1}}^{\infty}|R_{6,\tilde{j}n}|\left|\sum_{\tilde{k}\mid\tilde{j}}\tilde{\mu}(\tilde{k})\right|\leqslant\sum_{\tilde{j}=\tilde{1}}^{\infty}|R_{6,\tilde{j}n}|\sum_{\tilde{k}\mid\tilde{j}}|\tilde{\mu}(\tilde{k})|$$

$$\leqslant\sum_{\tilde{j}=\tilde{1}}^{\infty}|R_{6,\tilde{j}n}|\,\tilde{\tau}(\tilde{j})|\leqslant\sum_{j=1}^{\infty}|R_{6,jn}|\,\tau(j)$$

$$\leqslant\sum_{j=1}^{\infty}|R_{6,jn}|\,\tau(jn)\leqslant\sum_{j=1}^{\infty}|R_{6,j}|\,\tau(j)$$

The convergence of the last series follows from the equality $|R_{6,j}| = O(j^{-1-\varepsilon})$. We obtained

$$A_n = \sum_{\tilde{k}=\tilde{1}}^{\infty} \tilde{\mu}(\tilde{k})\, R_{6,\tilde{k}n}.$$

Then

$$f(x) = \sum_{n=0}^{\infty} A_n T_n(x) = A_0 + \sum_{n=1}^{\infty} \left(\sum_{\tilde{k}=\tilde{1}}^{\infty} \tilde{\mu}(\tilde{k})\, R_{6,\tilde{k}n} \right) T_n(x)$$

$$= A_0 + \sum_{j=1}^{\infty} R_{6,j} \sum_{\tilde{k}\mid j} \tilde{\mu}(\tilde{k})\, T_{j/k}(x).$$

Using again

$$f(1) = A_0 + \sum_{n=1}^{\infty} A_n = A_0 + \sum_{j=1}^{\infty} R_{6,j} \sum_{\tilde{k}\mid j} \tilde{\mu}(\tilde{k})$$

we finally get

$$f(x) = f(1) + \sum_{j=1}^{\infty} R_{6,j}(f) \sum_{\tilde{k}\mid j} \tilde{\mu}(\tilde{k})(T_{j/k}(x) - 1).$$

It is easy to observe that $E_n(f) = P_{2,n}(f)$, i.e. Theorem 4 includes as a special case Eterman's result. Theorem 5 also may be considered as a generalization of this result because of the following. The presented proof may be applied for each particular value of s for which $|\tilde{\mu}(\tilde{n})| \leqslant 1$. These values are $s = 2$, $s = 3$, $s = 4$, $s = 6$. It is not difficult to see as well that $E_n(f) = \frac{1}{2} R_{2,n}(f) = R_{4,n}(f)$. In the study of other cases one needs more precise estimation for the function $\tilde{\mu}(\tilde{n})$.

Acknowledgments

The author is grateful to Professor B. Bojanov for suggestions and guidance in preparing this paper.

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